Indian Statistical Institute

Midterm Exam. 2022-2023

Analysis II, B.Math First Year

Time : 3 Hours Date : 20.02.2023 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) You may freely apply any of the theorems we discussed in class.

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(1) (10 marks) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Prove that f is Riemann integrable if and only if for $\epsilon > 0$, there exist step functions s_1 and s_2 such that $s_1(x) \leq f(x) \leq s_2(x)$ for all $x \in [a,b]$, and

$$\int_a^b (s_2 - s_1) < \epsilon.$$

[A step function is a finite linear combination of indicator functions.]

(2) (10 marks) "There exists a sequence of continuous functions $\{f_n\}$ on $[0,1]$ such that $f_n \to 0$ pointwise but not uniformly on $[0,1]$". Is it true or false? Justify your answer.

(3) (20 marks) Let f be a Riemann integrable function on $[0,1]$, and suppose $\int_0^1 f^2 = 0$. If $c \in (0,1)$ and f is continuous at c, then compute the value of $f(c)$. Justify your answer.

(4) (20 marks) Let f and g be bounded functions on $[a,b]$ and suppose $f(x) = g(x)$ for all but finitely many $x \in [a,b]$. Prove that

$$\overline{\int_a^b} f = \overline{\int_a^b} g.$$

(5) (20 marks) Let $\{f_n\}$ be a sequence of continuous functions on (a,b) such that $f_n \to f$ uniformly on (a,b). If $\{x_n\} \subset (a,b)$ is a convergent sequence converges to a point in (a,b), then prove that

$$\lim_{n\to\infty} f_n(x_n) = f(\lim_{n\to\infty} x_n).$$

(6) (20 marks) Does the series of functions $\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}$ converge uniformly on $[0,1)$? Justify your answer.